Kang-Redner small-mass anomaly in cluster-cluster aggregation

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The large-time small-mass asymptotic behavior of the average mass distribution $\overline{P}(m,t)$ is studied in a d-dimensional system of diffusing aggregating particles for $1 \leqslant d \leqslant 2$. By means of both a renormalization group computation as well as a direct resummation of leading terms in the small-reaction-rate expansion of the average mass distribution, it is shown that $\overline{P}(m,t) \sim (1/t^d)(m^{1/d}/\sqrt{t})^{e_{KR}}$ for $m \ll t^{d/2}$, where $e_{KR} = \epsilon + O(\epsilon^2)$ and $\epsilon = 2 - d$. In two dimensions, it is shown that $\overline{P}(m,t) \sim \ln(m)\ln(t)/t^2$ for $m \ll t/\ln(t)$. Numerical simulations in two dimensions supporting the analytical results are also presented.

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I. INTRODUCTION

Reaction-diffusion systems in low dimensions provide an excellent testing ground for developing our understanding of the fluctuation effects in complex systems far from equilibrium. A great deal of information, both numerical and analytical, has been gathered over the past 20 years to show that the evolution of statistical properties in simple reaction-diffusion systems in low dimensions is anomalous, in the sense that it does not follow the corresponding mean field equations (see Ref. [1] for a short review). In low enough dimensions, in the instances where an exact solution is lacking, there are no formal methods by which the exponents characterizing the different physical quantities may be calculated. The renormalization group method (see Ref. [2] for a review) provides the only systematic way to calculate these exponents and thus understand fluctuation-dominated kinetics in reaction-diffusion systems. In this paper, we consider the model of irreversible aggregation of diffusing, massive particles $A_i + A_j \xrightarrow{\lambda} A_{i+j}$, in dimensions $1 \leqslant d \leqslant 2$, and use the renormalization group method to calculate the small-mass ($m \ll t^{d/2}$) behavior of $\overline{P}(m,t)$, the mean density of particles of mass m at time t. As we explain later in this section, this problem of determining $\overline{P}(m,t)$ requires using the full power of the renormalization-group method. This is unlike many other problems where just considering the rate equations with a renomalized reaction rate λ is enough to obtain the right answer.

We now give a more precise definition of the cluster-cluster aggregation (CCA) model $A_i + A_j \xrightarrow{\lambda} A_{i+j}$ and review earlier relevant results. Consider a d-dimensional lattice and particles that possess a positive mass. Given a configuration of particles on this lattice, the system evolves in time via the following microscopic moves: (i) With rate D, each particle hops to a nearest neighbor, and (ii) with rate λ, two particles on the same lattice site aggregate together to form a new particle whose mass is the sum of masses of the two constituent particles. As time increases, the number of particles decreases due to collisions and ultimately when $t \to \infty$, all particles coagulate together to form one massive aggregate. However, at finite times there is a well defined average mass distribution $\overline{P}(m,t)$, which is of interest to determine. It will be shown later [see the text following Eq. (16)] that the large time limit of this model is the same as the large-λ limit. The $\lambda \to \infty$ limit was studied numerically by Kang and Redner in one and two dimensions [3]. It was shown that $\overline{P}(m,t)$ has the scaling form $\overline{P}(m,t) = t^{-d} f(mt^{-d/2})$, where $d \leqslant 2$ is the dimension. The two exponents are easily determined from the two conditions $\int m\overline{P}(m)dm \sim t^0$ (mass conservation) and $\int \overline{P}(m)dm \sim t^{-d/2}$ (recurrence of random walks) [4]. The small-mass behavior of $\overline{P}(m,t)$ can be obtained by knowing the small-x behavior of the scaling function $f(x)$. On the basis of numerical simulations it was conjectured in Ref. [3] that $f(x) \sim x^{(2-d)/d}$. In one dimension, the model can be solved exactly [5,6]; it was shown that $f(x) \sim x$ or equivalently $\overline{P}(m,t) \sim mt^{-3/2}$ for $m \ll \sqrt{t}$. The one-dimensional solution uses the property of ordering of particles on a line and is not generalizable to higher dimensions. In two dimensions, $f(x)$ was seen numerically to increase with x for small x [3]. Also, in two dimensions, the scaling function could be determined in the limit of fixed x for $t \to \infty$, where $x = m\ln(t)/t$. In this case it was shown [7] that $\overline{P}(m,t) = t^{-2}\ln^2(t)e^{-x}$ for $x \ll \ln^{1/2}(t)$ and $|\ln(x)| \ll |\ln(t)|$. This result however, becomes incorrect in the limit when m is fixed as $t \to \infty$.

In this paper, we compute $\overline{P}(m,t)$ in $1 \leqslant d \leqslant 2$ in the limit $t \to \infty, m/m_0 = \text{fixed}$, where m_0 is the mass of the lightest particle at $t = 0$. We show that $\overline{P}(m,t) \sim (1/t^d)(m^{1/d}/\sqrt{t})^{e_{KR}}$ for $m \ll t^{d/2}$, where $e_{KR} = \epsilon + O(\epsilon^2)$ and $\epsilon = 2 - d$. In two dimensions, it is shown that $\overline{P}(m,t) \sim \ln(m)\ln(t)/t^2$ for $m \ll t/\ln(t)$. These results provide a theoretical basis to the results obtained by numerical methods in Ref. [3].

The CCA model may also be considered to be a special case of the more general model in which the aggregation kernel is mass dependent. For a review of results on the rate

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equation approach to this problem, see Refs. [8,9]. The dependence of $\overline{P}(m,t)$ on m in one dimension in this more general model has also been studied [10]. In this paper, we will restrict ourselves to the aggregation kernel which is mass independent; i.e., the rates λ and D are independent of mass. When the mass is ignored and $\lambda \to \infty$, the CCA model reduces to the well-studied $A+A \to A$ model [11]. The CCA model and its variants also find application in a large number of physical systems including colloidal suspensions [12], irreversible polymerization [13], aerosols and cloud formation [13], river networks [14], and coarsening phenomena [15].

Field theoretic methods have been previously used to study complex systems far from equilibrium (see Ref. [16] for a review). We briefly review results relevant to reaction-diffusion systems. In some earlier works [17,18], the effective reaction rate and the decay exponent of the average particle density were computed for the $A+A \to A(\varnothing)$ model. The renormalization group study of the same model with sources was done in Ref. [19]. In Ref. [20] the systematic renormalization group procedure for the computation of the average density and density-density correlation function in $kA \to \varnothing$ reaction was developed. In Ref. [21] renormalization group analysis of the $A+B \to \varnothing$ reaction in $d>2$ was used to study the effects of initial fluctuations on the late time decay of particle densities. The renormalization group technology developed by Peliti, and co-workers [18,20,21] was used to compute the average mass distribution of clusters in the CCA model in the intermediate-mass range in Ref. [7].

It turns out, however, that as far as the study of scaling properties of one-point correlation functions in most reaction-diffusion systems is concerned, the renormalization group is not a vital tool. Consider, for example, a single-species-annihilation model $A+A \to \varnothing$. Once the renormalization of the effective reaction rate is understood, the correct density decay exponent can be obtained from simple dimensional arguments [16]. Alternatively, one can use simple random walk arguments or the Smoluchowski approximation (we refer to the case in which the reaction rate is replaced by a time dependent reaction rate as the Smoluchowski approximation) [3,22] to obtain the correct values of decay exponents. A renormalized mean field theory or, alternatively, a version of Smoluchowski's theory can also be used to compute the average mass distribution in the cluster-cluster aggregation for intermediate masses [23].

In the CCA model considered in this paper, it turns out that the stochastic field $P(m,\vec{x},t)$, describing the continuous limit of the local mass distribution, has a nonzero anomalous dimension in $d<2$. The scaling exponent governing the dependence of the average mass distribution $\overline{P}(m,t)$ on mass is proportional to the anomalous dimension of the operator corresponding to the local mass distribution. As explained in Sec. II any approximation scheme which disregards this anomaly (such as the Smoluchowski approximation) predicts that $\overline{P}(m,t) \sim m^0$ when $m \ll t^{d/2}$, for any dimension. This is in contradiction with both the numerical results [3] as well as the exact result in one dimension [5,6]. The full power of renormalization group analysis has thus to be brought to bear in order to compute the anomalous dimension of $P(m,\vec{x},t)$ in the form of the $\epsilon = 2-d$ expansion. Since the peculiarity in the small-mass distribution of cluster-cluster aggregation was first discussed in Ref. [3] we refer to this as the *Kang-Redner* anomaly.

In addition to the calculation of $\overline{P}(m,t)$, we found many similarities between the problem of cluster-cluster aggregation and the problem of weak turbulence (see Ref. [24] for a review on weak turbulence). We elaborate on this connection in Sec. VI, where the Kang-Redner anomaly is interpreted as a violation of the Kolmogorov (constant flux) spectrum of particles in mass space due to strong flux fluctuations.

The rest of the paper is organized as follows. Section II contains a discussion of the stochastic integro-differential equation satisfied by $P(m,\vec{x},t)$. The mean field results, as well as the reasons for their failure in low dimensions are also included. In Sec. III, we analyze the large-time asymptotic behavior of $\overline{P}(m,t)$ in $d<2$ using the renormalization group method. We do the same for $d=2$ in Sec. IV. In Sec. V, we rederive the results of Secs. III and IV, using an explicit resummation of all diagrams giving the leading contribution to the average mass density in the limit of large time. Reasons for the failure of the Smoluchowski theory then become more transparent. In Sec. VI we elaborate on the connections with weak turbulence. Finally, we conclude with a summary and discussion of open problems in Sec. VII.

II. CONSTANT KERNEL CLUSTER-CLUSTER AGGREGATION AND MEAN FIELD ANALYSIS

The problem of computing density correlation functions in d-dimensional stochastic processes can be reformulated as an effective equilibrium problem in $(d+1)$ dimensions with the help of the Doi-Zel'dovich-Ovchinnikov trick [25]. One can then attempt to solve the problem using the powerful methods of statistical field theory, in particular, those based on renormalization group ideas. Starting from the lattice version of the CCA model, we would like to derive the corresponding field theory and the Langevin equation obeyed by $P(m,\vec{x},t)$, where $P(m,\vec{x},t)dmdV$ is the number of particles with masses in the interval $[m,m+dm]$ in the volume dV centered at $\vec{x}$. It was shown in Ref. [7] that the problem of finding all correlation functions of the local mass distribution is equivalent to finding all moments of the following stochastic integro-differential equation (stochastic Smoluchowski equation):

$$\left(\frac{\partial}{\partial t} - D\Delta\right) P(m,\vec{x},t) = \lambda P*P - 2\lambda N(\vec{x},t) P(m,\vec{x},t) + 2i\sqrt{\lambda}\,\xi(\vec{x},t) P(m,\vec{x},t), \qquad (1)$$

where $P*P = \int_0^m dm' P(m-m',\vec{x},t) P(m',\vec{x},t)$ is a convolution term, $\xi(\vec{x},t)$ is Gaussian white noise, and $N(\vec{x},t) = \int_0^\infty dm \overline{P}(m,\vec{x},t)$ is the local particle density. We are interested in $\overline{P}(m,t) = \langle P(m,\vec{x},t) \rangle$, where $\langle \cdots \rangle$ denotes averag-

ing with respect to the noise ξ. If the initial number of particles of different masses at different lattice sites are independent Poisson random variables parametrized by the initial average mass distribution $P_0(m)$, then the initial condition to be supplied with Eq. (1) is $P(\vec{x},m,0)=P_0(m)$. It is easy to check that Eq. (1) conserves the average mass density $\rho=\int m\overline{P}(m)dm$.

As particles aggregate, the typical mass grows in time as $t^{d/2}$. If we are interested in small masses, we need to consider masses smaller than the typical mass, $m \ll \rho(Dt)^{d/2}$. This can be achieved by considering m/m_0 to be fixed as $t\to\infty$, where m_0 is the smallest mass at $t=0$. In this case, the first term on the right hand side of Eq. (1) is almost surely small compared to the other terms. Consequently, the small-mass behavior of the local mass distribution is described by the following system of nonlinear stochastic partial differential equations:

$$\left(\frac{\partial}{\partial t}-D\Delta\right)P(m,\vec{x},t)=-2\lambda N(\vec{x},t)P(m,\vec{x},t) + 2i\sqrt{\lambda}\,\xi(\vec{x},t)P(m,\vec{x},t), \quad (2)$$

$$\left(\frac{\partial}{\partial t}-D\Delta\right)N(\vec{x},t)=-\lambda N^2(\vec{x},t)+2i\sqrt{\lambda}\,\xi(\vec{x},t)N(\vec{x},t). \quad (3)$$

Equations (2) and (3) demonstrate an interesting connection between this model and the $A+A\to A$ model. The stochastic field P can be identified with $\partial N/\partial N_0$, where N_0 is the initial density. Indeed, differentiating Eq. (3) with respect to N_0, and setting $P=\partial N/\partial N_0$, we obtain Eq. (2). Therefore, if the complete N_0 and t dependence of the mean density of particles is known, then so is the time dependence of the average mass distribution. But in practice, only the leading order time dependence of the mean density (which is independent of N_0) is known.

We are interested in the behavior of $\overline{P}(m,t)$ in the limit of fixed m and $t\to\infty$. We can then identify the particles with this fixed mass as B kind of particles and the remaining particles as A kind of particles. Then, clearly, the study of Eqs. (2) and (3) is equivalent to the study of a two-species reaction,

$$A+A\xrightarrow{\lambda}A, \qquad A+B\xrightarrow{\lambda}(\text{inert}), \quad (4)$$

in the limit when the concentration of B particles is much smaller than that of A particles. This two-species problem has been studied in $d=1$ for arbitrary diffusion rates [26]. Specializing the results of this paper to our case, we find that $\overline{P}(m,t)\sim t^{-3/2}$ for $t\to\infty$. Assuming that the large-time asymptotics of $\overline{P}(m,t)$ is universal, we can restore the m dependence using dimensional analysis, to obtain

$$\overline{P}(m,t)=C\frac{m}{\rho t^{3/2}}, \quad (5)$$

where C is a constant and ρ is the average mass density. Equation (5) matches with the exact results obtained for the CCA model in one dimension [5,6].

However, no exact solutions are available for the dimensions $d>1$. In the rest of the paper, we will be analyzing Eqs. (2) and (3) in $d>1$ using the dynamical renormalization group method. We will show that for small masses and $1 \leqslant d<2$,

$$\overline{P}(m,t)\sim\frac{1}{\rho(Dt)^d}\left(\frac{m}{\rho(Dt)^{d/2}}\right)^{e_{KR}}, \quad m\ll\rho(Dt)^{d/2}, \quad (6)$$

where $e_{KR}=\epsilon+O(\epsilon^2)$ and $\epsilon=2-d$. If $d=2$,

$$P(m,t)\sim\frac{1}{\rho(Dt)^2}\ln(t/t_0)\ln\left(\frac{m}{\rho Dt_0}\right)\left(1+\frac{1}{\ln(t/t_0)}\right) \quad \text{for} \quad \rho Dt_0\ll m\ll\rho Dt\ln(t/t_0). \quad (7)$$

Here, $t_0\sim\Delta^2/D$, where Δ is the lattice spacing.

Before doing the renormalization group analysis, let us analyze Eqs. (2) and (3) in the mean field (weak coupling) limit. Neglecting stochastic terms in the right hand side of Eqs. (2) and (3) and solving the resulting system of ordinary differential equations, we obtain

$$\overline{N}_{MF}(t)=\frac{N_0}{1+N_0\lambda t}, \quad (8)$$

$$\overline{P}_{MF}(t)=\frac{P_0}{(1+N_0\lambda t)^2}. \quad (9)$$

Thus, at large times, $\overline{P}(t)\sim t^{-2}$, given that mean field theory is applicable. Relative corrections to the mean field result are of the order of $g_0(t)=\lambda t/(Dt)^{d/2}$. Therefore mean field theory is asymptotically exact in $d>2$ [27] and diverges with t if $d<2$. Resummation of the most divergent terms in the weak coupling expansion of $\overline{P}$ around the classical solution is required and can be performed in the case at hand using the formalism of the renormalization group. The details of the computation are given in Sec. III. Here, we would like to demonstrate that accounting for the renormalization of the coupling constant alone does not yield the correct decay exponent as mentioned earlier. To the leading order in $\epsilon=2-d$, the renormalization of the effective reaction rate reduces to replacing λ in Eqs. (2) and (3) with the renormalized value $\lambda_R=f(\epsilon)t^{-\epsilon/2}$ and omitting stochastic terms (the renormalized mean field approximation). Eliminating $\overline{N}(t)$ from the resulting system of ordinary differential equations, one finds the following equation for $\overline{P}(m,t)$:

$$\frac{\partial\overline{P}}{\partial t}=-d\frac{\overline{P}}{t}. \quad (10)$$

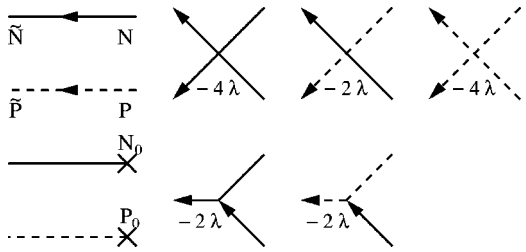

FIG. 1. Propagators and vertices of the effective theory, Eq. (12).

This implies that $\overline{P}(m,t) \sim m^0 t^{-d}$. In other words, $\overline{P}(m,t)$ scales with time according to its physical dimension. As a result it does not depend on mass. As we will show in Sec. III, arguments leading to this conclusion are incorrect, as they disregard the possibility of the anomalous dimension of the stochastic field P.

III. RENORMALIZATION GROUP ANALYSIS OF STOCHASTIC AGGREGATION

The average mass distribution $\overline{P}(t)$ and average particle density $\overline{N}(t)$ admit functional integral representations which can be obtained by applying the Martin-Siggia-Rose procedure [28] to Eqs. (2) and (3) [equivalently see Eqs. (2)–(4) of Ref. [7]]. We then obtain

$$\langle O(t)\rangle = \int \mathcal{D}N(\vec{x}',t')\mathcal{D}\tilde{N}(\vec{x}',t')\mathcal{D}P(\vec{x}',t')\mathcal{D}\tilde{P}(\vec{x}',t') \times O(t)e^{-S_{eff}[N,\tilde{N},P,\tilde{P}]}, \tag{11}$$

where $O(t)=N(\vec{x},t)$ or $P(\vec{x},t)$ and

$$S_{\rm eff} = \int_0^t d^dx\,d\tau[\tilde{N}(\dot{N}-D\Delta N)+\tilde{P}(\dot{P}-D\Delta P)+\lambda(\tilde{N}N^2 + 2\tilde{P}PN+\tilde{N}^2N^2+2\tilde{N}\tilde{P}NP+\tilde{P}^2P^2)] \tag{12}$$

is the effective action functional. Here $\tilde{P}$ and $\tilde{N}$ are the response fields. Perturbative expansions of $\overline{N}(t)$, $\overline{P}(m,t)$ in powers of λ can now be obtained in the standard way, see, for example, Ref. [29]. Feynman rules for constructing terms of these expansions are summarized in Fig. 1. Due to the nonrenormalization of the diffusion rate as well as the average mass density in the field theory, Eq. (12), in all that follows, we use units in which $\rho=D=1$. The average mass distribution $\overline{P}(m,t)$ is formally given by the sum of all diagrams built out of blocks presented in Fig. 1 with one outgoing punctuated line (P line). The contribution from each diagram is a function of $\lambda N_0 t$ and "bare" dimensionless reaction rate $g_0(t)=\lambda t/t^{d/2}$. Unless we are interested in the small-time expansion of $\overline{P}(m,t)$, $\lambda N_0 t$ cannot be treated as a small parameter. Therefore, the contributions of all diagrams proportional to a given power of $g_0(t)$ and various powers of $\lambda N_0 t$ have to be summed up. A simple combinatorial argument (see Ref. [20] for details) shows that the contribution of a diagram with n loops is proportional to $g_0(t)^n$. In the weak coupling regime the main contribution to $\overline{P}(m,t)$ and $\overline{N}(t)$ is, therefore, given by the sum of all tree diagrams, the first correction comes from summing all one-loop diagrams and so on. It turns out [20,30] that the sum of all tree diagrams gives the mean field answers, Eqs. (8) and (9). The expansion in powers of $g_0(t)$ is therefore the standard loop expansion around solutions of mean field equations. It is obvious that in $d<2$ the loop expansion is not very useful at large times as $\lim_{t\to\infty} g_0(t)=\infty$. Fortunately, the value of dimensionless reaction rate properly renormalized to account for the buildup of interparticle correlations turns out to be of order $\epsilon=2-d$ for large times. This allows one to convert the loop expansion into an ϵ expansion using the perturbative renormalization group method. Such an expansion works well for large times and will therefore yield all the information we need about the behavior of the average mass distribution in the strongly fluctuating regime. In computing loop corrections to any order, there are generally an infinite number of diagrams to sum. However these diagrams can be resummed in part if one introduces classical response functions G_{cl}^{NN} and G_{cl}^{PP}. Response function G_{cl}^{NN} (G_{cl}^{PP}) is equal to the sum of all tree diagrams with one outgoing and one ingoing line of types N (P). As was already mentioned, mean field densities Eqs. (8) and (9) are also equal to the sums of tree diagrams with one outgoing N or P line correspondingly. One then simply has to associate incoming lines with mean field densities and propagator lines with mean field response functions. Integral equations satisfied by classical densities and response functions are presented in diagrammatic form in Fig. 2.

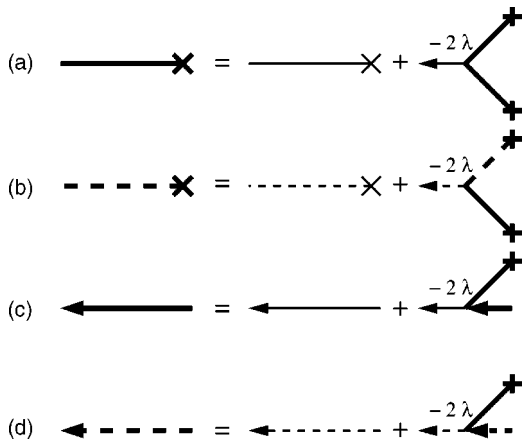

FIG. 2. Diagrammatic form of mean field equations for (a) average particle density, (b) average mass distribution, (c) NN-response function, and (d) PP-response function.

The solutions of the equations in Figs. 2(a) and 2(b) coincide with Eqs. (8) and (9), as they should. The equations in Figs. 2(c) and 2(d) can also be solved with the result

$$G_{cl}^{NN}(x_2,t_2;x_1,t_1) = G_{cl}^{PP}(x_2,t_2;x_1,t_1) = \left(\frac{N_0(t_2)}{N_0(t_1)}\right)^2 G_0(x_2-x_1,t_2-t_1), \tag{13}$$

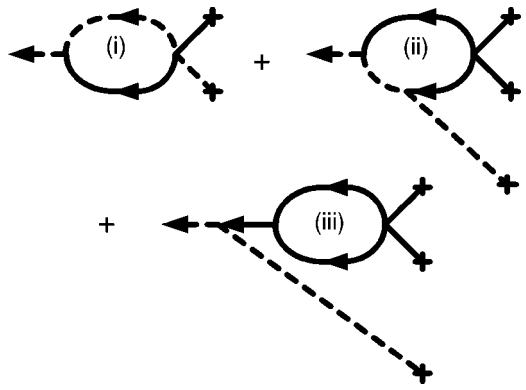

FIG. 3. One-loop corrections to the mean field answer for average mass distribution.

where G_0 is the Green's function of the linear diffusion equation.

Using the notion of mean field response functions and densities, one can easily classify all one-loop diagrams contributing to the average mass distribution. The result is presented in Fig. 3. A quick check shows that analytical expressions corresponding to diagrams (i) and (iii) containing primitive loops diverge in $d \geqslant 2$, which is consistent with the fact that the upper critical dimension of the effective theory Eq. (12) is 2. Computing the relevant integrals in $d=2-\epsilon$ dimensions we find the following one-loop expression for the average mass distribution:

$$\overline{P}(t)=\frac{P_0}{N_0^2}\left[\left(\frac{1}{\lambda t}\right)^2+\frac{1}{(8\pi)^{d/2}}\frac{1}{\lambda t^{(2-\epsilon/2)}}F(\epsilon)\right]\times\left[1+O\left(\frac{1}{\lambda N_0 t}\right)\right]+(\text{two}-\text{loop corrections}), \tag{14}$$

where $F(\epsilon)=(4/\epsilon)1-\epsilon/2/[(1+\epsilon/2)^2(1+\epsilon/4)]$. Equation (14) can be used to determine the large-time asymptotics of $\overline{P}(t)$ in the following way. The exact average mass distribution satisfies the Callan-Symanzik equation which we will derive below. Coefficients of this equation depend on the law of renormalization of all relevant couplings of the theory Eq. (12). One relevant coupling is the effective reaction rate. Its renormalization is known [18]. Below we will show that the only other relevant coupling is the initial mass distribution P_0. We will determine its renormalization law with one-loop precision, demanding that the expression Eq. (14) be nonsingular in the limit $\epsilon \to 0$ when expressed in terms of the renormalized coupling constant and renormalized initial mass distribution. This will determine the coefficients of the Callan-Symanzik equation up to terms of order g_R, where g_R is the renormalized dimensionless reaction rate. Solving this equation we will be able to compute the decay exponent of $\overline{P}(t)$ up to terms of order ϵ.

The dimensional analysis of effective vertex parts of the theory Eq. (12) shows that the only relevant *bulk* coupling is the effective reaction rate. Its relevance is due to the recurrence of random walks in $d \leqslant 2$. Reaction-rate renormalization accounts for all fluctuation effects in the $A+A \to A$ model.

However, the CCA model is more complicated and interesting due to the presence of *boundary*-relevant couplings. To identify them, we use the following version of dimensional analysis. Boundary coupling constants correspond to vertices with no incoming lines. As PP interactions can be neglected in the problem at hand, we are interested in boundary vertices with at most one P line. Assume for simplicity that $d=2$ (critical dimension). Assume also that the initial density $N_0=\infty$. This assumption is justified if one is interested in the large time-behavior of correlation functions in aggregation, as N_0 flows to infinity under renormalization group transformation to increasingly larger time scales, see Ref. [16] for more details. Let $\Gamma_{\alpha,\beta}(t)$, where $\alpha=0,1$; $\beta=0,1,2,\ldots$, be the simultaneous Green's function of the theory Eq. (12) with α P-lines and β N-lines with all external momenta equal to 0. Using Eq. (13), one can express $G_{\alpha,\beta}(t)$ in terms of corresponding vertex part $V_{\alpha,\beta}(t)$ as follows:

$$G_{\alpha,\beta}=\left(\frac{1}{t^2}\right)^{\alpha+\beta}\int_0^t d\tau \tau^{2(\alpha+\beta)}V_{\alpha,\beta}(\tau). \tag{15}$$

As the physical dimension of $G_{\alpha,\beta}(t)$ is $(L)^{-2\beta-4\alpha}$, where $V_{\alpha,\beta}(t) \sim t^{-\beta-2\alpha-1}$. As a result, Eq. (15) converges at small times for any α if $\beta>0$. If $\beta=0$ and $\alpha=1$, Eq. (15) diverges logarithmically. This divergence can be regularized using a small-time cutoff $1/\lambda N_0$ and leads to the renormalization of the initial mass distribution P_0. The latter plays the role of the coupling associated with $V_{1,0}$.

As a result of the discussed divergence, diagrams involving $V_{1,0}$ grow faster with time compared with diagrams with the same number of loops but with no subdiagrams contributing to renormalization of P_0. Consequently, diagrams belonging to the former class have to be resummed exactly in order to obtain the correct large-time behavior of the average mass distribution.

We conclude that fluctuations in the stochastic aggregation lead to two effects: renormalization of the effective reaction rate *and* renormalization of the initial mass distribution. It follows that the renormalization of these two couplings regularizes the perturbative expansion of $\overline{P}(m,t)$ to all orders.

As it turns out, the renormalization of P_0 is solely responsible for the Kang-Redner anomaly. In Sec. V we will analyze initial density renormalization by explicitly resumming small-time divergences in the perturbative expansion for $\overline{P}(m,t)$. Now, we will show how it appears formally within the framework of the perturbative renormalization group method. We follow dynamical renormalization group procedure described in Ref. [16].

Let us fix a reference time $t_0>0$. Expression (14) evaluated at t_0 is to be made finite by absorbing the divergences appearing as $\epsilon \to 0$ into the renormalization of the reaction rate and the initial density.

Let $g_0=\lambda t_0^{\epsilon/2}$ be the dimensionless "bare" reaction rate. As has been shown in Ref. [18], renormalized reaction rate g_R is related to g_0 via the following exact formula:

$$g_R=\frac{g_0}{1+\dfrac{g_0}{g^*}}. \tag{16}$$

Here $g^*=2\pi\epsilon[1+O(\epsilon)]$ is the nontrivial fixed point of the renormalization group flow in the space of effective couplings of Eq. (12). Recall that $g_0\sim\lambda$. Hence $\lim_{\lambda\to\infty}g_R=g^*$. It follows from the Callan-Symanzik equation that the large-time behavior of the $\overline{P}(m,t)$ is also determined by the fixed point value of the effective reaction rate g^*. We therefore conclude that the limits $t>0,\lambda\to\infty$ and $\lambda>0,t\to\infty$ of the Kang-Redner model belong to the same universality class as claimed in the Introduction.

Solving Eq. (16) with respect to g_0, expanding the result in powers of g_R, and substituting the expansion into Eq. (14), we obtain the average mass distribution at time t_0 as a power series in g_R:

$$\overline{P}(t_0)=\frac{P_0}{N_0^2 g_R^2 t_0^d}\left[1-\frac{1}{g^*}[1+O(g^*)]g_R+O(g_R^2)\right]. \tag{17}$$

As expected, the order-g_R term in Eq. (17) is still singular at $\epsilon=0$. To cancel the remaining divergences we have to introduce a renormalized initial mass distribution P_R:

$$P_R=Z(g_R,t_0,\epsilon)P_0, \tag{18}$$

where Z is a power series in g_R with coefficients chosen in such a way, that the average mass density

$$\overline{P}(t,g_R,P_R,t_0)=Z(g_R,t_0,\epsilon)\overline{P}(t,\lambda_0,P_0,\epsilon), \tag{19}$$

when expressed in terms of P_R,g_R, is nonsingular at $\epsilon=0$. Substituting Eq. (17) into Eq. (19) we find that

$$Z=1+\frac{g_R}{g^*}+O(g_R^2), \tag{20}$$

in order for $\overline{P}(t)$ to be nonsingular at the one-loop level.

Now we can derive the Callan-Symanzik equation. The fact that $\overline{P}(t,\lambda_0,P_0,\epsilon)$ does not depend on the reference time t_0 leads to the following equation for $\overline{P}(P_R)$:

$$t_0\frac{\partial}{\partial t_0}[Z^{-1}\overline{P}(P_R)]=0.$$

Noticing that $\overline{P}(t)=t_0^{-d}(P_R/N_0^2)\Phi(t/t_0,g_R)$, where Φ is a dimensionless function, one can convert the above condition into the Callan-Symanzik equation for $\overline{P}(t)$:

$$\left(t\frac{\partial}{\partial t}+\frac{1}{2}\beta(g_R)\frac{\partial}{\partial g_R}+d-\frac{1}{2}\gamma(g_R)\right)\overline{P}(t,g_R,P_R,t_0)=0, \tag{21}$$

where $\beta(g_R)=-2(\partial/\partial t_0)g_R$ is the beta function of the theory Eq. (12) and

$$\gamma(g_R)=-2\frac{1}{Z}t_0\frac{\partial}{\partial t_0}Z=-\frac{g_R}{2\pi}+O(g_R^2,g_R\ \epsilon) \tag{22}$$

is the gamma function.

It is well known that at large times and in $d<d_c=2$, solutions to Eq. (21) are governed by nontrivial fixed points (zeros) of the β function. Differentiating Eq. (16) with respect to t_0, one finds that $\beta(g_R)=g_R(g_R-g^*)$. Hence, $\beta(g_R)$ has a unique nontrivial fixed point $g_R=g^*$. It follows from the Callan-Symanzik equation [Eq. (21)] that $\overline{P}(t)\sim t^{-d^*}$, $t\to\infty$, where

$$d^*=d-\tfrac{1}{2}\gamma(g^*). \tag{23}$$

We see that the scaling dimension of $\overline{P}(m,t)$ differs from its physical dimension by a term proportional to the value of the γ function at the fixed point. This term is called the anomalous dimension of the field P. The physical reason for the presence of the anomalous dimension is the renormalization of the initial mass distribution by small time fluctuations.

As $g^*\sim\epsilon$, the substitution of the g_R expansion of γ into Eq. (23) yields the ϵ expansion for d^*. Using Eq. (22) we find that

$$d^*=d+\tfrac{1}{2}\epsilon+O(\epsilon^2). \tag{24}$$

Finally, one can restore the m dependence of $\overline{P}(m,t)$ using a dimensional argument. The result is

$$\overline{P}(m,t)=A(\epsilon)\frac{1}{t^d}\left(\frac{m^{1/d}}{\sqrt{t}}\right)^{e_{KR}}, \tag{25}$$

where $e_{KR}=\epsilon+O(\epsilon^2)$ is equal to twice the anomalous dimension of the stochastic field P.

Note that in $d=1$, $e_{KR}=1$ and $\overline{P}(m,t)\sim m^1$, which coincides with the exact answer [5]. However, at the moment we do not have reasons to believe, that higher order ϵ corrections to our answer for e_{KR} vanish identically for any $0<\epsilon\leqslant 1$. Equation (24) implies that the anomalous dimension of P vanishes at $d_c=2$. Yet, it follows from the general theory of the perturbative renormalization group that traces of the anomalous dependence of $P(m,t)$ on mass in $d<2$ must be present at the critical dimension as well. We analyze the Kang-Redner anomaly in $d=2$ in Sec. IV and compare renormalization group predictions with the conclusions of the direct numerical simulations of the system of diffusing-aggregating point particles on the two-dimensional lattice.

IV. KANG-REDNER ANOMALY IN TWO DIMENSIONS

It is well known that anomalies in $d_c-\epsilon$ dimensions lead to logarithmic corrections to the mean field theory answers in $d=d_c$. The Kang-Redner anomaly is not an exception. The small-mass behavior of the average mass distribution in two dimensions can be easily obtained by solving the Callan-Symanzik equation [see Eq. (21)]. Note that

$$\beta(g)|_{d=2}=\frac{1}{2\pi}g^2,$$
$$\gamma(g)|_{d=2}=-\frac{1}{2\pi}g[1+O(g^2)]. \tag{26}$$

Solving Eq. (21) with coefficients given by Eq. (26) and the initial condition

$$\overline{P}(t_0)=\text{const}\times\frac{1}{g_R^2 t_0^2}[1+O(g_R)] \tag{27}$$

produced by the mean field theory, we find that

$$\overline{P}(t)=\text{const}\times\frac{\ln(t/t_0)}{t^2}[1+O(1/\ln(t/t_0))]. \tag{28}$$

Such a behavior of $\overline{P}(m,t)|_{m=fixed}$ in two dimensions was originally seen by Kang and Redner in numerical simulations [3]. We have now shown that Eq. (28) can be obtained as a result of systematic renormalization group computation. Analyzing their model in two dimensions, Kang and Redner noticed that $\overline{P}(m,t)$ is a slowly varying (increasing, as suggested by Fig. 2 of Ref. [3]) function of mass. We can now quantify this observation by computing the mass dependence of the average mass distribution. In $d=d_c$, direct dimensional arguments cannot be used to restore the mass dependence of Eq. (28) due to the explicit dependence of $\overline{P}(m,t)$ on the lattice cutoff. Instead let us analyze Eq. (25) for $\overline{P}(m,t)$ in the limit $\epsilon\to+0$. Near $\epsilon=0$, the amplitude $A(\epsilon)\sim\epsilon^{-2}$. Expanding the right hand side of Eq. (25) in powers of ϵ and using the fact that $\overline{P}(m,t;\epsilon)$ is nonsingular at $\epsilon=0$ as a function of renormalized parameters, we find that

$$\overline{P}(m,t)=\frac{1}{t^2}\{C_1[\ln(t/t_0)]^2+C_2\ln(t/t_0)\ln(m/t_0)+C_3[\ln(m/t_0)]^2\}[1+O(1/\ln(t/t_0))]. \tag{29}$$

We are interested in $\overline{P}(m,t)$ in the limit $t\to\infty$, $m=$ fixed. Time dependence of $\overline{P}(m,t)$ is given by Eq. (28). Therefore, coefficient C_1 in Eq. (29) is 0. Hence

$$\overline{P}(m,t)=\text{const}\times\frac{\ln(t/t_0)\ln(m/t_0)}{t^2}[1+O(1/\ln(t/t_0))]. \tag{30}$$

Note that Eq. (30) is valid for $m\ll M(t)$, where $M(t)\sim t/\ln(t/t_0)$ is the typical mass. If $m\sim M(t)$, then $\overline{P}(m,t)\sim[\ln(t/t_0)/t]^2$, which coincides with the answer for $\overline{P}(m,t)|_{m\sim M(t)}$ in the intermediate mass range [7].

To check our analytical results, Eqs. (28) and (30), we did a numerical simulation of the system on a two-dimensional lattice. For the sake of computational efficiency, we chose to work in the limit $\lambda\to\infty$ so that a lattice site can hold utmost one particle. The lattice was chosen to be a square lattice of size 3000×3000 with periodic boundary conditions. In Fig. 4, we show the variation of $\overline{P}(m,t)$ with t for fixed small m. It is seen that $\overline{P}(m,t)=c(m)\ln(t)/t^2$, where $c(m)$ is some function of the mass m. This is in excellent agreement with Eq. (28). To determine $c(m)$, we studied the variation of $\overline{P}(m,t)/\overline{P}(1,t)$ with the mass m (see Fig. 5). We see that $c(m)\sim\ln(m)$, thus confirming Eq. (30).

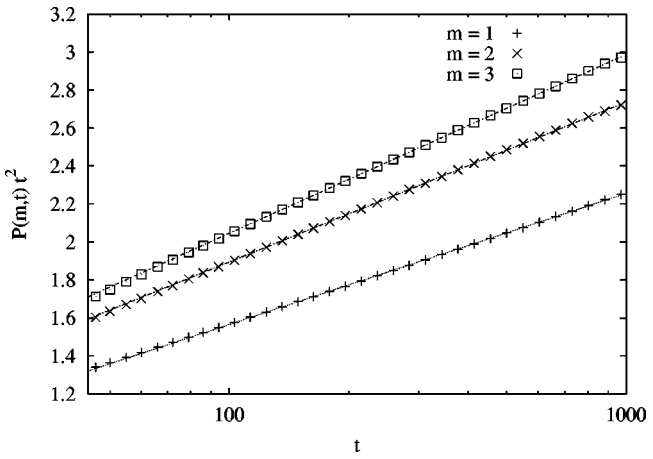

FIG. 4. The variation of $\overline{P}(m,t)t^2$ with t is shown for $m=1,2,$ and 4 on a semilog scale. The variation is linear, implying that $\overline{P}(m,t)=c(m)\ln(t)/t^2$, where $c(m)$ is some mass-dependent function.

V. KANG-REDNER ANOMALY VIA EXPLICIT RESUMMATION OF BOUNDARY DIVERGENCES

In this section we will derive Eq. (25) without using the formalism of the renormalization group. Instead, we will identify the principal set of diagrams contributing to the large-time limit of $\overline{P}(m,t)$ and derive a simple integral equation satisfied by the sum of these diagrams.

Additional divergences in the terms of the perturbative expansion of the average mass distribution discussed in Sec. IV are actually due to the quadratic singularity of the mean

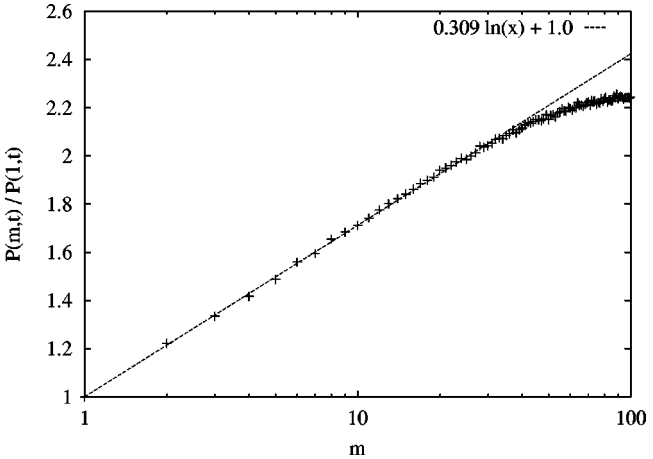

FIG. 5. The small-m behavior of $\overline{P}(m,t)/\overline{P}(1,t)$ is shown on a semilog plot. The variation is linear, implying that $c(m)\sim\ln(m)$ with $c(m)$ as defined in the caption of Fig. 4.

field expression Eq. (9) for the average mass distribution at $N_0=\infty$, $t=0$. Let us illustrate this statement with the example of diagram (i) of Fig. 3. Assuming for simplicity that $d=2$ and denoting ultraviolet cutoff by Δt, we find that the most divergent contribution coming from this diagram is

$$I_{(i)}\sim\frac{P_0}{2\pi\lambda^2N_0^2t^2}\ln\left(\frac{t}{\Delta t}\right)\int_0^t d\tau\frac{\lambda N_0}{1+\lambda N_0\tau}[1+O(1/\lambda N_0t)]. \tag{31}$$

Setting $N_0=\infty$ under the sign of integration is clearly impossible, as the resulting integral would diverge. Direct computation of Eq. (31) shows that

$$I_{(i)}\sim\frac{P_0}{2\pi\lambda^2N_0^2t^2}\ln\left(\frac{t}{\Delta t}\right)\ln(\lambda N_0t)[1+O(1/\lambda N_0t)]. \tag{32}$$

We see that $I_{(i)}$ is proportional to $\ln(t)^2$, not $\ln(t)$ as one would have deduced from the simple loop counting. The additional singularity is due to the fact that $\overline{P}_{MF}|_{N_0=\infty}\sim t^{-2}$. (Recall that $\overline{N}_{MF}|_{N_0=\infty}\sim t^{-1}$. As a result, divergences at $t=0$ are absent in the field theory of $A+A\to 0$ reaction.) The additional *boundary* singular terms are generally present in all Feynman integrals contributing to $\overline{P}(m,t)$ and have to be resummed to all orders of the perturbation theory to get the correct large-time asymptotics of the average mass distribution.

Let $\Pi(t_2,t_1)$ be the exact zero-momentum vertex function of type $\tilde{P}P$—the sum of all one-particle irreducible diagrams contributing to the $\tilde{P}P$ response function divided by the propagators corresponding to external lines. The Schwinger-Dyson equation for the exact average mass distribution reads

$$\overline{P}(t)=\overline{P}_{MF}(t)+\frac{1}{t^2}\int_0^t dt_2\int_0^{t_2}dt_1\Pi(t_2,t_1)\overline{P}(t_1). \tag{33}$$

Equation (33) is most easily derived using the formalism of Feynman diagrams. Its diagrammatic representation is given in Fig. 6(a). We know that in two dimensions, $\overline{P}(t)\sim t^{-2}$. Therefore, near $d=2$, the function $\eta(t)=\overline{P}(t)\lambda t^2$ is slowly varying. The following simplified version of Eq. (33) is therefore valid to the leading order in ϵ:

$$\eta(t)=1+\int_0^t dt_1\Pi(t_1)\,\eta(t_1), \tag{34}$$

where

$$\Pi(t)=t^2\int_0^t\frac{dt_1}{t_1^2}\Pi(t,t_1). \tag{35}$$

Differentiating Eq. (34) with respect to time, we find that $\eta(t)$ satisfies the following differential equation:

$$\frac{d\eta}{dt}(t)=\Pi(t)\,\eta(t). \tag{36}$$

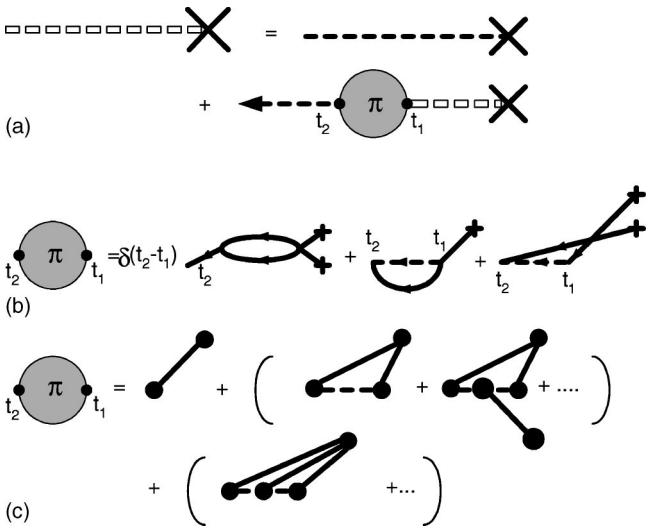

FIG. 6. (a) Schwinger-Dyson equation for $\Pi(t)$. Perturbative expansion of the polarization operator (b) in the number of loops, (c) in the order of exact vertex parts.

It turns out (see below) that at large times

$$\Pi(t)=\frac{1}{t}\left(\frac{\epsilon}{2}+O(\epsilon^2)\right)[1+O(1/t^{\epsilon/2})]. \tag{37}$$

Solving Eq. (36) with $\Pi(t)$ given by Eq. (37), we find that $\overline{P}(t)\equiv\eta(t)/t^2\sim t^{[-2+\epsilon/2+O(\epsilon^2)]}$. Thus, we have been able to rederive Eq. (24) without any reference to a renormalization group.

Before we turn to the derivation of Eq. (37), we would like to demonstrate that solving integral equation Eq. (34) is indeed equivalent to summing leading boundary singular terms in the perturbative expansion of $\overline{P}(t)$ to all orders. All divergences at $t=0,N_0=\infty$ can be regularized by setting the lower limit of integration in the integral in the right hand side of Eq. (34) equal to $1/\lambda N_0$. The resulting equation,

$$\eta(t)=1+\frac{\epsilon}{2}\int_{1/\lambda N_0t}^t\frac{dt_1}{t_1}\eta(t_1), \tag{38}$$

can be solved using the method of consecutive approximations. Setting $\eta(t)=\eta_0(t)+\eta_1(t)+\eta_2(t)+\cdots$ and treating the integral in the right hand side of Eq. (38) perturbatively, we find that $\eta_0(t)=1$ and

$$\eta_n(t)=\frac{1}{n!}\left(\frac{\epsilon}{2}\ln(\lambda N_0t)\right)^n,\quad n=1,2,\ldots. \tag{39}$$

Note that $\eta_1(t)$ is equal to the one-loop boundary singular term, $\eta_2(t)$ is the two-loop boundary singular term, and so on. Summing the resulting series for η one confirms our main result that $\eta(t)\sim t^{\epsilon/2}$. One loop-boundary singularity leads to mass-dependent logarithmic corrections to the average mass distribution. This is most easily seen by expanding Eq. (25) in powers of ϵ. These corrections, which were originally observed in Ref. [7], motivated to a certain extent the present investigation.

The knowledge of the first term in the ϵ expansion of the decay exponent of $\eta(t)$ leads to a good approximation of the average mass distribution, given that $\epsilon \ln(\lambda N_0 t) \sim 1$, but $\epsilon^2 \ln(\lambda N_0 t) \ll 1$. If the latter condition breaks down, we need to know the expansion of the polarization operator $\Pi(t)$ to second order in ϵ *and* modify Eq. (34) by including corrections proportional to the derivative $\eta(t)$ [which account for the fact that $\eta(t)$ is not constant].

We will now derive Eq. (37) by computing $\Pi(t)$ to the first order in ϵ using one-loop diagrams and verifying that higher-loop diagrams lead to terms of order ϵ^2 and higher. If the mean field theory for the average mass distribution was exact, operator Π would have been identically equal to zero. Consequently only loop diagrams contribute to $\Pi(t_2,t_1)$. The one-loop diagrams are shown in Fig. 6(b). The computation of corresponding Feynman integrals is straightforward. After integration with respect to t_1 we find their respective contributions to $\Pi(t)$:

$$\Pi_{(i)}(t) = \frac{-8\lambda t^{\epsilon/2}}{(8\pi)^{d/2} t} \frac{(1+2\epsilon)}{\epsilon(1+\epsilon/2)^2}, \tag{40}$$

$$\Pi_{(ii)}(t) = \frac{8\lambda t^{\epsilon/2}}{(8\pi)^{d/2} t} \frac{1}{\epsilon(1+\epsilon/2)}, \tag{41}$$

$$\Pi_{(iii)}(t) = \frac{-8\lambda t^{\epsilon/2}}{(8\pi)^{d/2} t} \frac{1}{(1+\epsilon/2)(2+\epsilon/2)}. \tag{42}$$

Note that individual contribution from diagrams (i) and (ii) are $1/\epsilon$ times bigger than the contribution from diagram (iii), which does not contain primitive loops. Yet, terms of order ϵ^{-1} cancel upon adding $\Pi_{(i)}(t)$ and $\Pi_{(ii)}(t)$ leaving terms of order up to ϵ^0—same as the leading order of terms of $\Pi_{(iii)}(t)$. This cancellation explains why we had to account for an apparently subleading contribution of diagram (iii) to the perturbative expansion of $\Pi(t)$. Such a cancellation is not accidental and happens at all orders of loop expansion: diagrams (i) and (ii) can be interpreted as the first two terms in the ϵ expansion of the first term in the cumulant expansion of $\Pi(t_2,t_1)$, see Fig. 6(c). This term corresponds to an exact total particle density $\bar{N}(t)$ connected to the $\tilde{P}P$ propagator via an exact $\tilde{P}NP$ vertex. The vertex is of the order of the fixed point coupling $g^* \sim \epsilon$, while the exact density is of the order ϵ^{-1} [20]. Therefore the contribution of the term in question to $\Pi(t_2,t_1)$ is of order 1, which is reflected in the cancellation of the terms of the lowest order in ϵ in every term of its loop expansion.

Adding together Eqs. (40)–(42), we find that

$$\Pi(t) = \frac{2\lambda t^{\epsilon/2}}{(8\pi)^{d/2} t}[1 + O(\epsilon)] + \text{two-loop corrections}. \tag{43}$$

Large-t behavior of $\Pi(t)$ can be obtained from Eq. (43) by replacing the bare reaction rate λ with the renormalized reaction rate $\lambda_R(t) \sim 2\pi\epsilon t^{-\epsilon/2}, t \to \infty$. The result does indeed coincide with Eq. (37).

It remains to verify that two- and higher-loop diagrams contribute only to higher order terms in the ϵ expansion of $\Pi(t)$. Order-ϵ^{-2} contributions from diagrams containing only primitive loops cancel as explained above. Order-ϵ^{-1} contributions from diagrams with two primitive loops and two-loop diagrams with one primitive loop are accounted for by the one-loop renormalization of the coupling constant in one-loop diagrams. Hence nontrivial corrections to the polarization operator come only from two-loop diagrams containing no primitive loops and nonsingular parts of all other two-loop diagrams. Simple counting shows that the contribution from such diagrams to $\Pi(t)$ is proportional to $[(\lambda_R(t) t^{\epsilon/2})^n]/\epsilon^{(n-2)} t \sim \epsilon^2/t$. A similar argument shows that n-loop diagrams contribute to $\Pi(t)$ at the order ϵ^n only.

Now it is very easy to characterize the class of diagrams giving the leading contribution to the ϵ expansion of $\Pi(t)$. The statistics of $N(t)$ is strongly non-Gaussian. Yet, the main contribution to the polarization operator $\Pi(t)$ comes from the diagrams proportional to the first and the second cumulants of the stochastic field $N(x,t)$ only. Non-Gaussian effects are due to the fact that these cumulants are connected to the $\bar{P}P$ propagator via exact vertices.

It is also possible to derive formula Eq. (28) without using the formalism of the renormalization group. Instead, one can solve Eq. (36) directly using the one-loop expression for the polarization operator $\Pi(t)$ in two dimensions. The latter can be obtained by setting in Eq. (43) $d=2$ and replacing the bare reaction rate λ with the renormalized reaction rate in two dimensions: $\lambda_R(t) = [4\pi/\ln(t/t_0)][1 + O(1/\ln(t/t_0))]$. The presented expression for $\lambda_R(t)$ is easy to compute by explicit resummation of all diagrams contributing to the renormalization of the bare reaction rate, see Ref. [20] for details. The resulting equation for $\eta(t) \equiv t^2 P(t)$ is

$$\dot{\eta}(t) = \frac{1}{t \ln(t/t_0)} \eta(t). \tag{44}$$

The solution is $\eta(t) \sim \ln(t/t_0)$. Correspondingly, $P(t) \sim \ln(t/t_0)/t^2$, which coincides with Eq. (28).

VI. KANG-REDNER ANOMALY AND CORRECTIONS TO THE KOLMOGOROV PARTICLE SPECTRUM

Now we will show that the mean field result $\bar{P}(m,t) \sim m^0$ can be interpreted as a constant flux (Kolmogorov) solution of the Smoluchowski equation. We will then interpret the Kang-Redner anomaly as a breakdown of Kolmogorov scaling due to strong flux fluctuations developing at large times.

Averaging Eqs. (2) and (3) with respect to noise one gets the following relation between one- and two-point mass distribution functions:

$$\frac{\partial \langle P \rangle}{\partial t} = -\lambda \langle PN \rangle, \tag{45}$$

$$\frac{\partial \langle N \rangle}{\partial t} = -\frac{1}{2} \lambda \langle N^2 \rangle. \tag{46}$$

Let us look for solutions of Eq. (46) having the form

$$\overline{P}(m,t)=\frac{\overline{N}(t)}{M(t)}J(\mu,t), \tag{47}$$

where $\mu=m/M(t)$. Recall that $M(t)=\overline{N}^{-1}$ is the typical mass. The new dependent variable J has a simple physical meaning: $\int_0^\mu d\mu' J(\mu',t)$ is the average number of particles with masses less than $M(t)\mu$ contained in the volume $\overline{N}^{-1}(t)$.

At times much less than $t_c=(\lambda)^{-2/\epsilon}$, the relative fluctuations of local density are small. As a result, mean field theory is applicable and $\langle JN\rangle\approx\langle J\rangle\langle N\rangle$. As a result, Eq. (45) simplifies to

$$\frac{\partial}{\partial t}\left(\frac{J}{\mu}\right)=\frac{\partial J}{\partial \mu}. \tag{48}$$

Therefore, J/μ is a locally conserved quantity with flux equal to $(-J)$. Note that $J>0$. Therefore the cascade of J/μ is *inverse* in the terminology of turbulence: its flux is directed towards the small masses. We see that self-similar solutions of Eq. (46) correspond to constant flux solutions of Eq. (48). The latter is just $J=$const. Constant flux solutions of kinetic equations are called Kolmogorov solutions in the theory of weak turbulence, see Ref. [24] for details. We therefore conclude that in the mean field approximation

$$\langle P\rangle(m,t)=\frac{\overline{N}(t)}{M(t)}\mu^{e_{kolm}}, \tag{49}$$

where $e_{kolm}=0$ is the exponent, which determines the Kolmogorov scaling of the average mass distribution. Note that the flux J also has a meaning of dimensionless particle density (the number of particles in the volume $\overline{N}^{-1}$), which is an obvious integral of motion.

The fact that $J=$const means that particles are equipartitioned between the system's degrees of freedom and particles' flux is identically equal to zero. The characteristic feature of the state Eq. (49) of our system is therefore the presence of nonzero constant flux of one integral of motion and the equipartition of the other. Similar kind of behavior has been observed in models of turbulent advection [31]

We know, however, that the mean field approximation is invalid in the limit of large times, if the dimension is or less because of strong fluctuations of local particle density. Using results of the previous sections one can interpret the Kang-Redner anomaly as the anomaly in the constant flux condition:

$$\mu\frac{\partial J}{\partial \mu}=\frac{e_{KR}}{d}J, \tag{50}$$

where e_{KR} is the Kang-Redner exponent.

Solving Eq. (50), we find that $P(m,t)\sim\mu^e$, where $e=e_{kolm}+e_{KR}/d$. Therefore, the Kang-Redner anomaly can be also interpreted as a correction to the Kolmogorov scaling of the average mass distribution due to strong fluctuations.

VII. CONCLUSION

In the present paper we have shown that the problem of cluster-cluster aggregation in $d\leqslant 2$ can be effectively analyzed using the renormalization group method. We have demonstrated that the dependence of the average mass distribution on mass is determined by the anomalous dimension of the stochastic field P (local mass distribution). This anomaly is due to the relevance of boundary $(t=0)$ fluctuations for the large-time asymptotics of $\overline{P}(m,t)$. In that respect the phenomenon of the Kang-Redner anomaly resembles the phenomenon of boundary phase transition in equilibrium statistical mechanics, see Ref. [2] for a review. Formally, the anomalous dimension of the local mass distribution is a consequence of the nontriviality of the γ function of the effective field theory Eq. (12). The fact that $\gamma(g)\neq 0$ is ultimately responsible for the breakdown of the Smoluchowski theory (or equivalently, the renormalized mean field theory) is applied to the model at hand. At present, renormalization group analysis seems to be the only theoretical method of studying the problem of cluster-cluster aggregation in $d>1$. This is not quite satisfactory, as the analysis is essentially perturbative in nature. However, our theoretical predictions concerning the behavior of the average mass distributions at small masses in two dimensions have been unambiguously confirmed numerically.

In Sec. VI we have shown that there is a relation between cluster-cluster aggregation and the theory of weak turbulence. In particular, we demonstrated that the Kang-Redner anomaly can be interpreted as a correction to the Kolmogorov spectrum of particles in mass space. However, we must stress that the analogy between our model and the phenomenon of turbulence must be taken with a pinch of salt: while the cascade of the conserved quantity in our model happens along the mass axis only, conserved quantities (such as energy or enstrophy) in more traditional turbulent systems flow through the scales of the physical space.

The method of dynamical renormalization group developed in the context of the model at hand can be applied to other nonequilibrium particle systems as well. In particular, the problem of cluster-cluster aggregation with annihilation of particles can be solved using techniques similar to those outlined in this paper [32]. This latter problem is related to the computation of the domain wall persistence exponent for the one-dimensional q-state Potts model [32].

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